PMM U.S.S.R.,Vol.52,No.5,pp.593-602,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

NON-AXISYMMETRIC SUBMERGED JETS*

N.I. YAVORSKII

The generalized multipole approach developed in /1/ for axisymmetric, non-selfsimilar submerged jets is extended to the non-axisymmetric case. In the case of jets with large asymmetry, a periodic rotation of the direction of asymmetry with distance from the jet source by a right angle, is predicted. It is shown that the first three terms of the asymptotic expansion at the point at infinity obtained in /1/ for axisymmetric jets remain valid in the non-axisymmetric case. The influence of the asymmetry on the stability of the jet flow and the problems of the convergence of the expansions obtained in the axisymmetric case are discussed. The possibility of formulating the problem in a bounded region is indicated and its relation to the stability of the flow is shown.

1. The basic results of /1/ are represented by the following assertions.

The asymptotic behaviour of non-selfsimilar jets is completely determined by the characteristic solutions $\mathbf{w}, q$ of the Navier-Stokes equations, linearized on the exact selfsimilar solution $\mathbf{v}_1$ obtained by Landau in /2/

$$(\mathbf{w}, \nabla)\mathbf{v}_1 + (\mathbf{v}_1, \nabla)w = -\nabla q + \nu\Delta\mathbf{w}, \ \operatorname{div} \mathbf{w} = 0 \tag{1.1}$$

and the eigenfunctions (multipoles) depend exponentially on the spherical radius

$$\mathbf{w} = \mathbf{W}(\theta)R^{-\alpha}, \ q = Q(\theta)R^{-\alpha-1}$$

where (R, θ, φ) are spherical coordinates. The exponent α is determined as the eigenvalue of the corresponding spectral problem on $\mathbf{W}, Q$ which follows from (1.1).

The first three terms of the expansion of stream function ψ have the form

$$\psi = \nu R y_1(x) + \nu \ln R u(x) + \nu z(x) + \ldots, \ x = \cos\theta \tag{1.2}$$

$$y_1(x) = 2\frac{1-x^2}{A-x}, \quad A > 1 \tag{1.3}$$

$$u(x) = B(1-x^2)\frac{1-Ax}{A(A-x)^2}, \quad z(x) = z_1(x) + c_0 u(x) \tag{1.4}$$

where $y_1(x)$ is the solution due to Landau /2/ and $z_1(x)$ is the analytic solution of the linear inhomogeneous equation given in /1/. The unknown constants A, B, c_0 are obtained by specifying the basic integrals of conservation of momentum, flow rate and transverse component of the angular momentum respectively.

We can determine with the same accuracy the peripheral velocity, i.e.

$$w_\varphi = c_1\Gamma_2(x)(1-x^2)^{-1/2}R^{-2} + \ldots. \tag{1.5}$$

where $\Gamma_2(x) = (1-x^2)(A-x)^{-2}$, and we can find the arbitrary constant c_1 by specifying the longitudinal component of the angular momentum.

We find that the same multipolar approach can also be used in the case of a non-axisymmetric problem of non-selfsimilar submerged jets.

In this case the solution of (1.1) can be written in the form

$$\mathbf{w} = \sum_{n=1,\, m=0}^{\infty} [\mathbf{W}_{1nm}(x)\cos m\varphi + \mathbf{W}_{2nm}(x)\sin m\varphi]\, R^{-\alpha_{nm}} \tag{1.6}$$

$$q = \nu \sum_{n=1,\, m=0}^{\infty} [Q_{1nm}(x)\cos m\varphi + Q_{2nm}(x)\sin m\varphi]\, R^{-\alpha_{nm}-1}$$

Let

$$\mathbf{w} = (1-x^2)^{m/2}\left[U(x)\cos m\varphi, \frac{V(x)}{\sqrt{1-x^2}}\cos m\varphi, \frac{W(x)}{\sqrt{1-x^2}}\sin m\varphi\right]R^{-\alpha} \tag{1.7}$$

*Prikl.Matem.Mekhan.,52,5,760-772,1988

$$q = \nu (1 - x^2)^{m/2} Q(x) R^{-\alpha-1}$$

For simplicity we have omitted the indices and the summation sign in (1.7). Formally, (1.7) resembles the representation of the adjoint Legendre polynomials, and its form is governed by the form of the right-hand side of (1.1), so that the functions $U(x)$, $V(x)$, $W(x)$, $Q(x)$ would be analytic with α chosen in the corresponding manner. Substituting (1.7) into system (1.1), written in a spherical system of coordinates, and eliminating the quantity $(1 - x^2)V''$ with help of the equation of continuity, we arrive at the following system of equations:

$$\begin{aligned}
&(1-x^2)U'' = 2(m+1)U' + [m(m+1) - (\alpha-1)(\alpha-2)]U - \\
&\quad (\alpha+1)Q + y_1U' + \left[(\alpha+1)y_1' - \frac{mxy_1}{1-x^2}\right]U + \left(y_1'' + \frac{2y_1}{1-x^2}\right)V \\
&(1-x^2)V' = (1-x^2)(2-\alpha)U + mxV + mW \\
&(1-x^2)W'' = (2m+2\alpha-4)U + 2V' + (y_1+2mx)W' + \\
&\quad \left[(\alpha-1)y_1' - \frac{mxy_1}{1-x^2} + m(m-1) - \alpha(\alpha-1)\right]W - mQ \\
&(1-x^2)Q' = \alpha(1-x^2)U' - 2mxU + (y_1+mx)V' + \\
&\quad \left[\alpha y_1' + (2-m)\frac{xy_1}{1-x^2} + m^2 - \alpha(\alpha-1)\right]V - mW' + mxQ \\
&-1 \leqslant x \leqslant 1
\end{aligned} \tag{1.8}$$

We could say that Eqs.(1.8) represent the result of the action of a non-selfconjugate generalized differential Legendre operator /4/, and therefore analytic solutions exists in the neighbourhood of the singularities $x = \pm 1$, and the conditions of the boundedness of U'', V', W'', Q' are equivalent to the demand that the functions be analytic at these points. The conditions of analyticity follow from (1.8) in which the function $y_1(x)$ is represented in the form (1.3) and $x = 1$ or $x = -1$. Choosing the parameter α suitably, we can continue the solution analytically from the point $x = 1$ up to the point $x = -1$ (or vice versa).

We see from (1.8) that the conditions of analyticity represent four homogeneous equations with seven unknowns U, U', V, V', W, W', Q. Three of them can be specified arbitrarily (and it is convenient to choose U, W, Q). This agrees completely with the physical formulation of the problem of non-selfsimilar jet flow outside a sphere of radius R_0 on which an arbitrary continuous velocity field v_R, v_ϑ, v_φ is specified, naturally under the condition that the systems of eigenfunctions $\{U_{nm}\}_{n=1}^\infty$, $\{V_{nm}\}_{n=1}^\infty$, $\{W_{nm}\}_{n=1}^\infty$ corresponding to the eigenvalues α_{nm} are complete for every asimuthal number $m = 0, 1, 2, \ldots, \infty$.

The latter assertion can be proved for the case when $\mathrm{Re} = 0$ $A(=\infty)$ where the Reynolds number Re is obtained in terms of the total momentum of the jet J /2/ and depends monotonically on the constant A from the solution (1.3), and $\mathrm{Re} \to \infty$ as $A \to +1$:

$$\mathrm{Re} = \left(\frac{J}{\pi\rho\nu^2}\right)^{1/2}, \quad J = 16\pi\rho\nu^2 A\left[1 + \frac{4}{3(A^2-1)} - \frac{A}{2}\ln\frac{A+1}{A-1}\right]$$

Indeed, seeking the solutions of (1.8) in the form of the polynomials in x, U, Q in the n-th degree, V in the $(n+1)$-th degree and W in the $(n+2)$-th degree we find the following spectral values:

$$\begin{aligned}
&\alpha_1 = n+m+2, \ \alpha_2 = n+m+2, \ \alpha_3 = n+m \\
&\alpha_4 = -n-m-1, \ \alpha_5 = -n-m-1, \ \alpha_6 = -n-m+1
\end{aligned} \tag{1.9}$$

The eigenvalues α_1, α_2, α_3 correspond to the problem of the flow outside the sphere, and α_4, α_5, α_6 correspond to the internal problem. Using the exponents $\alpha_1, \ldots, \alpha_6$ we can construct the solutions in a spherical layer or in any other doubly connected region bounded by star-like surfaces. Thus when $\mathrm{Re} = 0$, the eigenvalues are integral and the corresponding eigenfunctions are polynomials. From (1.9) we see that for every $m = 0, 1, 2, \ldots$ the family of eigenfunctions forms a basis of the space of all polynomials whose completeness in $C[-1, 1]$ is well-known.

When $\mathrm{Re} > 0$, the study of the problems of completeness of the eigenfunctions becomes very difficult, since the eigenvalues α occur in Eqs.(1.8) in a non-linear manner. There is still no rigorous mathematical theory embracing such spectral problems, so we shall restrict ourselves to qualitative arguments supported by appropriate numerical results.

The coefficients of system (1.8) and the momentum $J(A)$ are analytic with respect to the parameter $0 \leqslant 1/A < 1$. From this we can naturally predict, by analogy with the known theorems on the parametric dependence of solutions of the systems of ordinary differential equations, a continuous dependence (piecewise differentiable at $0 < \mathrm{Re} < \infty$) of the eigenfunctions on Re, regarded as a parameter for $0 \leqslant \mathrm{Re} < \infty$. In this case the number of eigenfunctions and their linear independence will be preserved at least in some neighbourhood of the point $\mathrm{Re} = 0$, and we will have no reason to assume that the completeness of the systems of eigenfunctions will be lost in this neighbourhood when Re increases from zero. It should be noted that the eigenvalues α_{nm} will also become functions of the Reynolds number and both the eigenfunctions and eigenvalues may take complex values. Numerical calculations confirm the continuous (and possibly piecewise differentiable) relationship $\alpha_{nm}(\mathrm{Re})$ (Fig.1-4) together with the corresponding eigenfunctions $U_{nm}(x, \mathrm{Re})$, $V_{nm}(x, \mathrm{Re})$, $W_{nm}(x, \mathrm{Re})$, $Q_{nm}(x, \mathrm{Re})$.

The Runge-Kutta-Meerson method was used to carry out the computations with a relative accuracy of 10^{-5}, according to the following scheme. Starting from the singularities $x = \pm 1$ of system (1.8) of sixth-order equations, we construct the triads of linearly independent solutions $\mathbf{U}_1^{\pm}(x)$, $\mathbf{U}_2^{\pm}(x)$, $\mathbf{U}_3^{\pm}(x)$ where 6 is the vector $\mathbf{U} = [U, U', V, W, W', Q]^T$, up to some matching point x_c, $-1 < x_c < 1$ situated within the region of the steepest gradients of the function required. The condition of analyticity of the solution is

$$c_1^+\mathbf{U}_1^+(x_c) + c_2^+\mathbf{U}_2^+(x_c) + c_3^+\mathbf{U}_3^+(x_c) = c_1^-\mathbf{U}_1^-(x_c) + c_2^-\mathbf{U}_2^-(x_c) + c_3^+\mathbf{U}_3^-(x_c)$$

The necessary condition for a non-trivial solution $c_k^{\pm}$ $(k = 1, 2, 3)$ of this system to exist is, that the sixth-order determinant

$$\Delta(\alpha, \mathrm{Re}, m) = |\,\mathbf{U}_1^+\mathbf{U}_2^+\mathbf{U}_3^+\mathbf{U}_1^-\mathbf{U}_2^-\mathbf{U}_3^-\,|$$

should vanish.

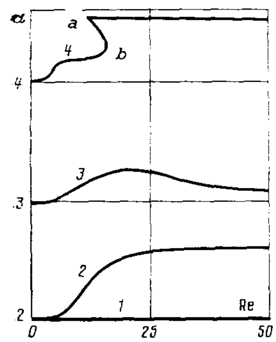

Fig.1

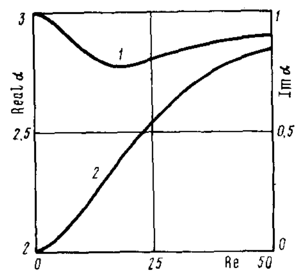

Fig.2

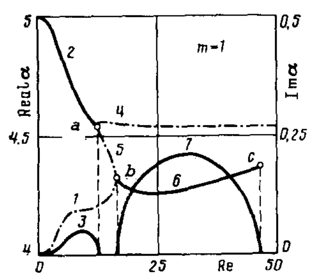

Fig.3

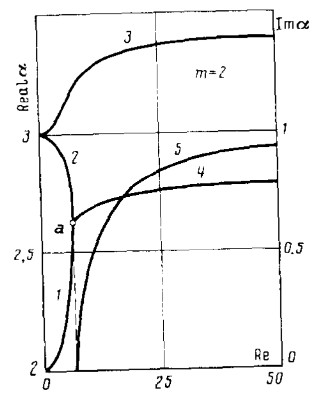

Fig.4

The equation yields the spectral values $\alpha_n(\mathrm{Re}, m)$. We see that the eigenvalues do not depend on x_c, since the matching of solutions at the point x_c produces an analytic solution everywhere within the region $-1 \leqslant x \leqslant 1$.

It should be noted that when $\mathrm{Re} = 0$, it follows from (1.9) that for every m the spectrum is $\alpha = m, m+1, m+2, \ldots$ and $\alpha = m$ is the singly, while $\alpha = m+2, m+3, \ldots$ are triply degenerate eigenvalues. When $\alpha = m+1$, $m > 0$, the solution is doubly degenerate, although one might have deduced from (1.9) that no degeneracy exists. The "supplementary" solution is a solution of the form $U = Q \equiv 0$, $V(x)$ is a zero-order polynomial and $W(x)$ of the first order, and this was not taken into account when constructing the solutions in the form of polynomials, and in deriving (1.9). When $\alpha > 0$, no other solutions exist. When $\mathrm{Re} > 0$, the degeneracy is removed and some of the eigenvalues remain real, while others form complex conjugate pairs.

Fig.1 shows the real exponents for the case $m = 1$ as a function of the Reynolds number. The straight line *1* which corresponds to $\alpha(\mathrm{Re}) = 2$ and curve *2*, from two real branches generated by the double eigenvalue $\alpha = 2$ at $\mathrm{Re} = 0$. Curve *3* represents a unique real branch of a triply degenerate eigenvalue $\alpha = 3$ at $\mathrm{Re} = 0$, whose other two branches form a complex conjugate pair (Fig.2). The exponent $\alpha(\mathrm{Re})$ shown by curve *4* has two cusps *a* and *b*, and a more complex origin (see Fig.3).

Curve *1* in Fig.2 ($m = 1$) represents the real part, and curve *2* the imaginary part of the eigenvalue of the complex conjugate pair corresponding to $\alpha = 3$ at $\mathrm{Re} = 0$.

Fig.3 ($m = 1$) demonstrates the fairly complex changes undergone by the exponent as the Reynolds number increases. The dot-dash line shows the real eigenvalue corresponding to curve *4* in Fig.1. When $\mathrm{Re} = 0$, one real branch (curve *1*) and two complex conjugate branches (not shown in the figure) depart from the point $\alpha = 4$. Similarly, when $\mathrm{Re} = 0$, one real branch (not shown in the figure) departs from the point $\alpha = 5$, together with two complex conjugate branches whose real part is represented by curve *2*, and imaginary part by curve *3*. The complex conjugate pair merges at point a generating two real exponents (curves *4* and *5*). At point *b* two real exponents (curves *1* and *5*) merge and a complex conjugate pair is generated whose real part is described by curve *6* and imaginary part by curve *7*. At point *c* the complex pair is transformed into a real pair of exponents (not shown in the figure).

Fig.4 shows the relations $\alpha(\mathrm{Re})$ for the azimuthal number $m = 2$. Curve *1* corresponds to the real exponent emerging from the non-degenerate eigenvalue $\alpha = 2$ at $\mathrm{Re} = 0$. Curves *2* and *3* represent real branches of the double eigenvalue $\alpha = 3$ at $\mathrm{Re} = 0$. At point *a* the two real exponents merge (curves *1* and *2*) and a complex conjugate pair is generated whose real part is described by curve *4* and imaginary part by curve *5*.

The presence of complex exponents of the spherical radius R in expansion (1.6) is important for understanding a number of physical effects (see Sect.3).

2. The velocity profile in the exact selfsimilar formulation of the laminar submerged jet should have the form /5/

$$\mathbf{v} = \nu R^{-1}\mathbf{V}(\theta, \varphi)$$

The Landau solution (1.3) refers to a class of flows in which $\mathbf{V} = \mathbf{V}(\theta)$. The problem arises of the existence of asymptotically non-axisymmetric submerged jets when $\mathbf{V} = \mathbf{V}(\theta, \varphi)$. Such solutions are admissible in the linear approximation to the non-axisymmetric case. Indeed, when $m = 1$, (1.9) implies that Eqs.(1.8) have a solution when $\alpha = 1$ and $\mathrm{Re} = 0$ $(n = 0)$. An exact solution of (1.8) for $\alpha = 1$, $m = 1$ and any Re can be found by assuming that the functions U, V, W, Q are polynomials with respect to the variable $1/(A - x)$. The required solution is

$$U(x) = \frac{2}{(A-x)^2}, \quad V(x) = \frac{1}{(A-x)^2} - \frac{A}{A^2-1}\frac{1}{A-x} \tag{2.1}$$

$$W(x) = -\frac{1}{A^2-1}\frac{1}{A-x}, \quad Q(x) = \frac{4}{(A-x)^3} - \frac{2A}{A^2-1}\frac{1}{(A-x)^2}$$

It can be shown that solution (2.1) is consistent in the sense that the next approximation obtained by iteration over the non-linearity has a solution. Solution (1.3) corresponds to the case when the z axis coincides with the axis of the jet.

Let the axis of the jet not coincide with the z axis and be directed along the unit vector $\boldsymbol{\tau} = (\sin\theta_0\cos\varphi_0,\ \sin\theta_0\sin\varphi_0,\ \cos\theta_0)$. We denote the unit vector of the spherical system of coordinates $O(R, \theta, \varphi)$ constructed relative to the Cartesian system of coordinates x, y, z by $\mathbf{n}_R, \mathbf{n}_\theta, \mathbf{n}_\varphi$, and by $\mathbf{n}_R', \mathbf{n}_\theta', \mathbf{n}_\varphi'$ the unit vectors of the spherical system of coordinates $O'(R, \theta', \varphi')$ constructed relative to the Cartesian system of coordinates x', y', z' whose z' axis is directed along $\boldsymbol{\tau}$ and whose origin of coordinates coincides with the point O. In this case we have

$$\mathbf{n}_R = \mathbf{n}_R', \quad (\mathbf{n}_\theta, \mathbf{n}_\theta') = -\cos\psi, \quad (\mathbf{n}_\theta, \mathbf{n}_\varphi') = \sin\psi \tag{2.2}$$

Using Landau's results (1.3) /2/, we obtain the exact solution of the Navier-Stokes equations in the $O(R, \theta, \varphi)$ system of coordinates

$$v_R = \frac{2\nu}{R}\left[\frac{A^2-1}{(A-\cos\theta')^2} - 1\right] \tag{2.3}$$

$$v_\theta = \frac{2\nu}{R}\frac{\sin\theta'}{A-\cos\theta'}\cos\psi, \quad v_\varphi = -\frac{2\nu}{R}\frac{\sin\theta'}{A-\cos\theta'}\sin\psi$$

It can be shown that the following relations hold:

$$\cos\theta' = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\varphi - \varphi_0) \tag{2.4}$$

$$\cos\theta_0 = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\psi$$

and they can be used to obtain the solution (2.3) in the $O(R, \theta, \varphi)$ system of coordinates.

In the case when $\theta_0 \ll 1$, i.e. when the amount of axial asymmetry is small, we have from (2.3) and (2.4)

$$\theta' = \theta - \theta_0 \cos(\varphi - \varphi_0), \quad \psi = \theta_0 \frac{\sin(\varphi - \varphi_0)}{\sin\theta} \tag{2.5}$$

$$w_R = \frac{4\nu}{R}\theta_0 \frac{A^2 - 1}{(A - \cos\theta)^3} \sin\theta \cos(\varphi - \varphi_0)$$

$$w_\theta = \frac{2\nu}{R}\theta_0 \left[\frac{A^2 - 1}{(A - \cos\theta)^2} - \frac{A}{A - \cos\theta}\right] \cos(\varphi - \varphi_0)$$

$$w_\varphi = -\frac{2\nu}{R}\theta_0 \frac{\sin(\varphi - \varphi_0)}{A - \cos\theta}$$

which agrees, apart from the numerical multiplier $2(A^2 - 1)\theta_0$, with solution (2.1). From this we can conclude that the Landau solution represents, in fact, a unique principal term of the asymptotic expansion in the non-axisymmetric formulation of the problem also. A solution of Eqs.(1.8) at $\alpha = 2$, $m = 1$ and all values of the Reynolds number can be obtained in the same manner from the asymptotic representation (1.2)-(1.4) written in the $O'(R, \theta', \varphi')$ system of coordinates as $\theta_0 \to 0$. Thus we see from the results of the computations given in Sect.1, that the contribution of the non-axisymmetric additions is given by the terms of the expansion with exponents $\alpha > 2$.

Thus when $R \to \infty$, the jet becomes axisymmetric and the asymptotic representation of the solution (1.2)-(1.4) remains valid in the non-axisymmetric case also.

3. When the solution is given in the form (1.6), it does not satisfy the complete Navier-Stokes equations. We can use the expansions (1.6), as was done in /1/, to construct a general solution of the Navier-Stokes equations (under the condition that the set of eigenfunctions $\{\mathbf{w}, q\}$ is complete in $C[-1, 1]$ $(0 \leqslant \theta \leqslant \pi)$, and this is assumed), if we carry out the expansion over a more complete set of exponents. This set will include, in particular, the powers which appear when (1.6) is substituted into the non-linear terms. Such a family of powers must have a group property, namely that the linear and non-linear terms yield exponents belonging to the same family. Additional terms of the expansion appear as solutions of the linear inhomogeneous equations whose right-hand side is determined from the known non-linear terms.

The solution can be written in the form (the summation is carried out in n and m, from $n = 1$, $m = 0$ to $n = \infty, m = \infty$)

$$v_R = \nu \sum (1 - x^2)^{m/2} R^{-\mu_{nm}} \mathbf{\Phi}(m\varphi)\, \mathbf{U}_{nm} \mathbf{\Phi}^T(\nu_{nm} R) \tag{3.1}$$

$$v_\theta = \nu \sum (1 - x^2)^{(m-1)/2} R^{-\mu_{nm}} \mathbf{\Phi}(m\varphi)\, \mathbf{V}_{nm} \mathbf{\Phi}^T(\nu_{nm} R)$$

$$v_\varphi = \nu \sum (1 - x^2)^{(m-1)/2} R^{-\mu_{nm}} \mathbf{\Phi}\left(m\left(\varphi - \frac{\pi}{2}\right)\right) \mathbf{W}_{nm} \mathbf{\Phi}^T(\nu_{nm} R)$$

$$\frac{p}{\rho} = \nu^2 \sum (1 - x^2)^{m/2} R^{-\mu_{nm}-1} \mathbf{\Phi}(m\varphi)\, \mathbf{Q}_{nm} \mathbf{\Phi}^T(\nu_{nm} R)$$

Here

$$\mu_{nm} + i\nu_{nm} = 1 + \sum_{j=2}^{\infty} n_j (\alpha_{jm_j} - 1), \quad \sum_{j=2}^{\infty} m_j = m \tag{3.2}$$

$$\mathbf{U}_{nm} = \begin{bmatrix} u_{nm}^{11} & u_{nm}^{12} \\ u_{nm}^{21} & u_{nm}^{22} \end{bmatrix} + \sum_{k=1}^{\infty} \frac{\ln^k R}{R^k} \begin{bmatrix} u_{knm}^{11} & u_{knm}^{12} \\ u_{knm}^{21} & u_{knm}^{22} \end{bmatrix} \tag{3.3}$$

$$\mathbf{\Phi}(\beta) = (\cos\beta, \sin\beta)$$

n_j, m_i are non-negative integers chosen so that the inequalities $\mu_{nm} < \mu_{n+1m}$ hold, the matrices $\mathbf{V}_{nm}, \mathbf{W}_{nm}, \mathbf{Q}_{nm}$ are represented in a form analogous to that of the matrix $\mathbf{U}_{nm}$ (3.3), and their elements are functions of x.

The appearance of complex exponents (3.2) in the non-axisymmetric case (3.1) represents an important difference between this case and the axisymmetric case /1/ of the problem of a non-selfsimilar submerged jet.

One of the physical consequences of this is, that in the case of a strongly asymmetric jet terms with $m \neq 0$ will play a major role in the initial segment. In particular, in the problem of a jet flowing out of a rectangular opening whose ratio of adjacent sides differs strongly from unity, the terms in question will be the terms with $m = 1$. The presence of complex exponents will lead, firstly, to the appearance of oscillations in the velocity profile, and secondly, as R (or z) increases, the jet will contract with a period $2\pi/\mathrm{Im}\,\alpha_1$ in the direction of the greatest asymmetry (the larger side of the rectangle) and this can be interpreted, on a background of the general widening of the jet flow, as a periodic rotation of the jet about its axis by 90°. When the value of R is increased further the velocity profile will be smoothed out and will tend to become selfsimilar.

What we said above refers to laminar jets. However, the qualitative -type deductions made above can also be related to the turbulent jets since the Bussinesq hypothesis of turbulent viscosity /6, 7/ can be applied to them. In order to compare the experimental data with the results obtained, it is sufficient to take the experimental value of the turbulent Reynolds number $\mathrm{Re}_T = 35$ /8/. Experiments /8, 9/ show fairly good qualitative agreement ($z/d \sim 20$ and more) with the characteristic features of the flow shown above for the case of "rectangular" jets, and the photograph included in /9/ shows that the period of rotation of the jet is of the order of the width of the jet in the initial cross-section, which agrees with the data given in Figs.2 and 4 where $\mathrm{Im}\,\alpha \sim 1$.

Another consequence is related to the problem of the hydrodynamic stability of the jet flow. As we have already shown, the contribution of the asymmetry can lead, at sufficiently high intensities of asymmetric multipoles, to oscillations in the velocity profile. The Rayleigh theorem on inviscid hydrodynamic instability at points of inflection of the velcoity profile for one-dimensional plane flows is well-known. In the three-dimensional case we have its analogue for axisymmetric plane parallel flows. In the general case the criterion of hydrodynamic instability loses the Rayleigh formulation, but a change in sign of the velocity derivative whose numerical value is not small, can serve as the source of instability in this case. From Fig.2 it follows that complex exponents appear when $\mathrm{Re} = 0$, and we can conclude that strongly asymmetric jets, as well as the classical displacement layer, lose stability when the Reynolds number becomes vanishingly small. If on the other hand the asymmetry is "moderate", the critical Reynolds number should depend on the degree of asymmetry.

Thus laminar jets are unstable under fairly strong perturbations and the domain of applicability of the solution (3.1)-(3.3) is sufficiently limited (when $\mathrm{Re} \simeq 15$, the jets are unstable under infinitely small perturbations /10/), although, as has already been shown, the proposed generalized multipole approach also finds application in the study of developed turbulent jet flows.

4. We have shown in Sect.1 that the solution of the problem of jet flow represented as an expansion in terms of the characteristic hydrodynamic multipoles can be applied to a boundary value problem in a two-phase region bounded by star-like surfaces, and in particular in a spherical layer, although in this case we must bring in the characteristic solutions corresponding to the exponents α_{nm} with $\mathrm{Real}\,\alpha_{nm} < 0$. As was done in Sect.3, we can formally construct a general solution, and in the present case it will be in the form of expansion (3.1) in which the index n varies from $-\infty$ to ∞, and $\mu_{nm} < 0$ if $n < 0$. If the expansion converges and shows the required degree of smoothness, it will represent a solution of the stationary Navier-Stokes equations for the boundary value problem of jet flows of very general form.

Let us briefly consider the problems of the convergence of series (3.1). Since the series in question are power series, in the case of the hydrodynamic problem for the outside of a sphere of radius R_0, the series $(n > 0)$ will converge everywhere in the region $R > R_0$ provided that they converge when $R = R_0$. Thus, if using the representations (3.1) we find that it is possible to satisfy the boundary conditions on the sphere $R = R_0$, the representations in question will be the solutions of the Navier-Stokes equations in the region in question. It remains to clarify the convergence of the series $R = R_0$. In what follows, we shall assume for simplicity that the flow has axial symmetry $(m = 0)$, although analogous results can be obtained in the asymmetric case also. The solution can be conveniently written in terms of the stream function /1/

$$\psi = \nu \sum_{n=-\infty}^{\infty} R^{2-\mu_n}\left[y_n(x) + \sum_{k=1}^{\infty} u_{kn}(x)\frac{\ln^k R}{R^k}\right] \tag{4.1}$$

$$p = \rho\nu^2 \sum_{n=-\infty}^{\infty} R^{-1-\mu_n}\left(g_n(x) + \sum_{k=1}^{\infty} h_{kn}(x)\frac{\ln^k R}{R^k}\right] \tag{4.2}$$

$$\mu_n = 1 + \sum_{j=-\infty}^{\infty} [n_j(\alpha_j - 1) + m_j(\gamma_j - 1)];\ \sum_{j=-\infty}^{\infty} n_j \geqslant 1,\ \ \sum_{j=-\infty}^{\infty} m_j \geqslant 2 \tag{4.3}$$

$$v_\varphi = \nu \sum_{n=-\infty}^{\infty} \frac{R^{-\zeta_n}}{\sqrt{1-x^2}}\left[\Gamma_n(x) + \sum_{k=1}^{\infty}\Omega_{kn}(x)\frac{\ln^k R}{R^k}\right] \tag{4.4}$$

$$\zeta_n = 1 + \sum_{j=-\infty}^{\infty} [n_j(\alpha_j - 1) + m_j(\gamma_j - 1)] \tag{4.5}$$

$$\sum_{j=-\infty}^{\infty} n_j \geqslant 0,\ \ \sum_{j=-\infty}^{\infty} m_j \geqslant 1;\ \ n_j, m_j = 0, 1, 2, \ldots$$

((4.3) differs from (4.5) in the constraints imposed on the sums n_j and m_j), where α_j and γ_j are the eigenvalues of the corresponding spectral problems formulated in /1/. The exponents are ordered, as in Sect.1, $\mathrm{Real}\,\mu_n \leqslant \mathrm{Real}\,\mu_{n+1}$, $\mathrm{Real}\,\zeta_n \leqslant \mathrm{Real}\,\zeta_{n+1}$. Let A_n, B_n, M_n be the intensities of the corresponding multipoles and $|A_n|, |B_n|, |M_n| < c / n^\alpha$, $\alpha > 1$, $0 < c < \infty$, and we shall assume for simplicity that the flow rate is zero. (The case with non-zero flow rate, i.e. when we have terms containing $\ln R$, can be dealt with in the same manner). We must estimate the contribution of the terms appearing as a result of quadratic non-linearity. The terms are proportional to

$$A_{i_1} \ldots A_{i_s} B_{j_1} \ldots B_{j_p} M_{k_1} \ldots M_{k_m} R^{1-\alpha_{i_1}} \ldots R^{1-\alpha_{i_s}} R^{1-\alpha_{j_1}} \ldots$$
$$\ldots R^{1-\alpha_{j_p}} R^{1-\gamma_{k_1}} \ldots R^{1-\gamma_{k_m}}$$

and if the sum $\mathrm{Real}\,\{(\alpha_{i_1}-1)+\ldots+(\gamma_{k_m}-1)\}$ lies in the interval $[N, N+1)$, their contribution when $N \gg 1$ can be estimated using the quantity $c_1/(1+N)^{\alpha}$, $c < c_1 < \infty$, $c_1 = \mathrm{const}$ as the upper limit (see the appendix). From this it follows that the series in question converge when $R \geqslant R_0$ if the assumptions are sufficiently natural and general.

It should be noted that when a hydrodynamic problem in a spherical layer or in any other similar region is considered, the convergence of series (4.1)-(4.5) becomes open to question. If, however, the intensities of the multipoles corresponding to the exponents with negative real parts decay exponentially rapidly $\sim (R_1/R_0)^{-|N|} / |N|^{\alpha}$ as $|N|$ increases, then (see the appendix) the complete series will in this case converge. The actual calculation of the multipole intensities from the boundary conditions leads to an infinite non-linear algebraic system which can be solved by the recurrence method, and to do this we must orthogonalize, one after the other, the systems of functions y_n, u_{kn}, Γ_n, Ω_{kn}. We may find the process quite time consuming, should the approximate numerical solution demand a fairly large number of eigenfunctions. We find, however, that the contribution of the terms appearing as the result of iterations over the non-linear terms is asymptotically small as $N \to \infty$ (see the appendix; it is therefore sufficient to confine ourselves to the solution of the linear problem for large N. This considerably simplifies the algebraic system for determining A_n, B_n, M_n and indicates that it is well-posed.

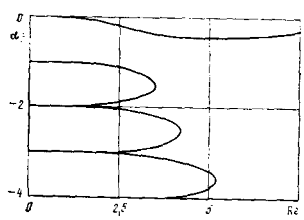

Fig.5

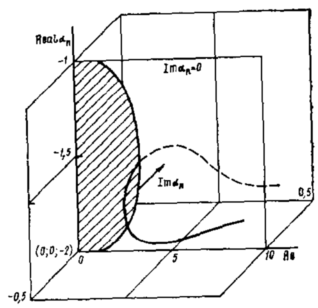

Fig.6

It should be noted that the choice of the basis in the form of the eigenfunctions is not the only possible one for a problem in a bounded region. We can, in principle, consider only the Galerkin-type approximations. It would appear however that the most preferable basis is that of characteristic multipoles, since it has a clear physical meaning for every term of the expansion and this is important when the problem is solved numerically. It is very likely that the characteristic basis offers the best convergence. In a number of cases it is sufficient to consider one or two terms of the expansion in order to obtain sufficient physical information concerning the solution /11/.

The characteristic feature of the axisymmetric solution is that the exponents with $j > 0$, i.e., with a positive real part, are real $\mathrm{Im}\,\alpha_j = \mathrm{Im}\,\gamma_j = 0$ /11/. In the case when $j < 0$, the indices α_j will be real when $0 \leqslant \mathrm{Re} \leqslant \mathrm{Re}_*$, and merge successively when $\mathrm{Re} \geqslant \mathrm{Re}_* = 3.5$ (Fig.5) forming complex conjugate pairs (Fig.6). This indicates a qualitative rearrangement of the flow and its possible instability (see Sect.1). It is interesting to note that the value of the critical Reynolds number at which the instability of a circular jet has been observed experimentally $\mathrm{Re}_{cr} = 3.7 - 4.1$ /12/ is very close to the value Re_*. At the same time the value of the critical Reynolds number found earlier /10, 13/ using the usual methods of the theory of hydrodynamic stability is much greater than the experimental value.

Appendix. Let us assess the contribution of the terms of expansion (4.1)-(4.5) with exponents μ_n, ζ_n lying within the interval $(N, N+1)$, for sufficiently large N, for the problem of jet flow outside the sphere $R \geqslant R_0$.

Let us assume, for simplicity, that the exponents α_n, γ_n are integers $(\alpha_n = \gamma_n = n)$, $n > 0$, and the flow rate Q is zero, the latter indicated by the corresponding absence of terms with $\ln R$ from the expansions (4.1)-(4.5). Then the interval shown will contain only a single value $\mu_N = N$ (if α_n, γ_n are fractional or complex, the estimate obtained below will not be changed significantly, and this will be mentioned at the appropriate time). In this case the expansions for the velocity and pressure fields will take the form

$$\mathbf{v} = \sum_{n=1}^{\infty} \mathbf{v}_n, \qquad p = \sum_{n=1}^{\infty} p_n \quad (\rho = 1) \tag{A.1}$$

where

$$\begin{aligned} &\mathbf{v}_n = \mathbf{v}_n{}^0 + \mathbf{u}_n, \quad p_n = p_n{}^0 + q_n \\ &\mathbf{v}_n = \mathbf{f}_n(\cos\theta) R^{-n}, \quad \mathbf{v}_n{}^0 = \mathbf{f}_n{}^0(\cos\theta) R^{-n} \\ &p_n = g_n(\cos\theta) R^{-n-1}, \quad p_n{}^0 = g_n{}^0(\cos\theta) R^{-n-1} \end{aligned} \tag{A.2}$$

satisfy the equations

$$\begin{aligned} &\frac{1}{\mathrm{Re}} \Delta \mathbf{v}_n{}^0 = \nabla p_n{}^0 + (\mathbf{v}_1, \nabla)\mathbf{v}_n{}^0 + (\mathbf{v}_n{}^0, \nabla)\mathbf{v}_1 \\ &\frac{1}{\mathrm{Re}} \Delta \mathbf{u}_n = \nabla q_n + \sum_{k=2}^{n-1} (\mathbf{v}_k, \nabla)\mathbf{v}_{n+1-k} \end{aligned} \tag{A.3}$$

The velocity and pressure coordinates are referred to their characteristic values on the sphere $R = R_0$. From (A2) and (A3) we find

$$\begin{aligned} &v_{n,i} = v^0_{n,i} + \sum_{j,k=1}^{3} L_{ijk} \sum_{l=2}^{n-1} v_{l,j} v_{n+1-l,k} \\ &L_{ijk} = \mathrm{Re}\, \Delta^{-1} \left(\delta_{ij} \frac{\partial}{\partial x_x} - \frac{\partial}{\partial x_i} \Delta^{-1} \frac{\partial^2}{\partial x_j \partial x_k} \right) \end{aligned} \tag{A.4}$$

where Δ^{-1} is the inverse Laplace operator. Since the operator L_{ijk} is integral, it can be shown that it is bounded in the space of functions $\mathbf{v}_n$ (A2) where $f_{n,i}, f_{n,i}{}^0 \in L_2([-1, 1])$

$$\| L_{ijk} \| \leqslant 1/3\lambda\, \mathrm{Re}\, R_0 < \infty; \; i, j, k = 1, 2, 3; \; 0 < \lambda < \infty \tag{A.5}$$

and the norm is given by the relations

$$\| \mathbf{v}_n \| = R_0^{-n} \| f_n \|, \quad \| \mathbf{f}_n \| = \sum_{i=1}^{3} \| f_{n,i} \|_{L_2} \tag{A.6}$$

From (A4) we conclude, taking (A5) and (A6) into account, that

$$\| \mathbf{f}_n \| \leqslant \| \mathbf{f}_n{}^0 \| + \lambda\, \mathrm{Re} \sum_{l=2}^{n-1} \| \mathbf{f}_l \| \| \mathbf{f}_{n+1-l} \| \tag{A.7}$$

Putting

$$f(n) = \| \mathbf{f}_n \|, \quad \| \mathbf{f}_n{}^0 \| \leqslant c_0/n^{\alpha}, \quad \alpha > 1 \tag{A.8}$$

we can write, for $n \gg 1$, the relation (A7) in integral form where n takes real values

$$f(n) \leqslant c_0/n^{\alpha} + \lambda\, \mathrm{Re} \int_2^{n-1} dx\, f(x) f(n+1-x) \tag{A.9}$$

The change to an integral relation is even more justified, since as $n \to \infty$, the exponents μ_n, ζ_n constructed using the non-integral α_j, γ_j densely fill the interval $[N, N+1), N \gg 1$ with a bounded distribution function, and this implies that condition (A9) will also hold for the fractional exponents. In the case of complex exponents the argument n will represent the real part of the exponent (the imaginary part leads to the appearance of oscillating multipliers not exceeding unity in modulus, which can be replaced by unity in the course of constructing the norm).

Expression (A9) written as an equality, can be used as the equation for determining the upper limit of the function $f(n)$ for large n, and this is wholly sufficient to solve the problems of the convergence of the expansions. Indeed, let us strengthen the inequality (A9)

$$f(x) < f_0(x) + \mu \int_1^x dy f(y) f(x+1-y) \tag{A.10}$$

$$f_0(x) = cx^{-\alpha}, \; c_0 < c < \infty, \; \alpha > 1, \; x \geqslant 1, \; \mu = \lambda\, \mathrm{Re} > 0$$

and we have, by definition, $f(x) \geqslant 0, \; f(1) > 0$.

Let a continuous solution $g(x)$ exist of the equation

$$g(x) = f_0(x) + \mu \int_1^x dy g(y) g(x+1-y) \tag{A.11}$$

It can be shown that $g(x) > 0$. Let us write

$$u(x) = f(x) - g(x) \tag{A.12}$$

From (A10)-(A12) it follows that

$$u(x) < \mu \int_1^x dy u(y) [f(x+1-y) + g(x+1-y)] \quad \text{(A.13)}$$

Therefore $u(1) < 0$. Since the function $u(x)$ is continuous, a number $a > 1$ exists such that $u(x) < 0$ for $1 \leqslant x < a$. Let $u(a) = 0$. Then (A13) will lead to a contradiction

$$\int_1^a dy u(y) [f(a+1-y) + g(a+1-y)] > 0$$

when $u(x) < 0$, $f(a+1-x) + g(a+1-x) \geqslant 0$, $1 \leqslant x < a$. Thus $u(x) < 0$ for all finite x.

Thus

$$f(x) < g(x), \quad 1 \leqslant x < \infty \quad \text{(A.14)}$$

Let us write

$$\varphi(x-1) = g(x), \quad \varphi_0(x-1) = f_0(x) \quad \text{(A.15)}$$

Then Eq.(A11) can be written in the form

$$\varphi(x) = \varphi_0(x) + \mu \int_0^x dy \varphi(y) \varphi(x-y) \quad \text{(A.16)}$$

We can solve Eq.(A16) using the Laplace transformation

$$\Phi(\lambda) = \Phi_0(\lambda) + \mu \Phi^2(\lambda) \quad \text{(A.17)}$$

$$\Phi_{1,2} = \frac{1}{2\mu} [1 \pm \sqrt{1 - 4\mu\Phi_0}]$$

$$\Phi(\lambda) = \int_0^\infty e^{-\lambda x} \varphi(x)\, dx, \quad \Phi_0(\lambda) = \int_0^\infty d^{-\lambda x} \varphi_0(x)\, dx$$

The quantity $\varphi_0(x) = c/(1+x)^\alpha$; therefore we have /14/

$$\Phi_0(\lambda) = c\, \Gamma(1-\alpha, \lambda)\, \lambda^{\alpha-1} e^\lambda \quad \text{(A.18)}$$

In order to find the asymptotic formula for $\varphi(x)$ as $x \to \infty$, it is sufficient to investigate the solutions (A17) as $\lambda \to 0$. In this case

$$\Phi_0(\lambda) = c\Gamma(1-\alpha)\, \lambda^{\alpha-1} + o(\lambda^{\alpha-1})$$

and two roots of (A17) have the form

$$\Phi_1 = \mu^{-1} - \Phi_0(\lambda) + O(\Phi_0^2), \quad \Phi_2 = \Phi_0(\lambda) + O(\Phi_0^2)$$

Applying the inverse Laplace transformation we see that as $x \to \infty$, the first root is

$$\varphi_1(x) = \mu^{-1}\delta(x) - \varphi_0(x) + o(\varphi_0) \quad \text{(A.19)}$$

and the second root is

$$\varphi_2(x) = \varphi_0(x) + o(\varphi_0) \quad \text{(A.20)}$$

It follows from (A19) that for sufficiently large x the quantity $\varphi_1(x) < 0$, and this is unacceptable for the norm. Thus the principal term of the asymptotic expansion is given, as $x \to \infty$, by formula (A20) and is independent of μ. Since $\mu = \lambda\,\mathrm{Re}$, we can conclude that the convergence of the series (4.1)-(4.5) is independent of the value of the Reynolds number Re. Nevertheless, we must remember that the intensities of characteristic multipoles depend on Re, and the constants c_0, α in the estimate $f_0(N) \leqslant c_0 N^{-\alpha}$ may, in general, depend on the number Re. Therefore, we cannot eliminate the possibilty of the existence of such an Re for which the conditions $\alpha > 1$ or $c_0 < \infty$ might be violated. On the other hand, the principal asymptotic term of (A20) indicates that for large values of N the solution is determined by the contribution of the solution of the uniform problem, while the contribution of the part of the solution obtained by iteration over the non-linearities is vanishingly small.

It can be shown that the estimates discussed above will hold for a very wide class of velocity profiles close to the selfsimilar Landau solution. The characteristic asymptotic behaviour of the terms of the expansion described above may make the proposed method of constructing the solutions of the Navier-Stokes equations suitable from the practical, computational point of view, although the question of whether the proposed expansions are the best of all possible expansions remains open.

It should be noted that in the case of the problem of jet flow in a spherical layer we can obtain analogous assertions concerning the convergence of the series (4.1)-(4.5). In particular, the inequality (A10) will be transformed, in the case of a solution on the sphere $R = R_0$, to the form

$$f(x) < f_0(x) + \mu \int_{-\infty}^{\infty} dy f(y) f(x+1-y) \quad \text{(A.21)}$$

We can also show that

$$f(x) < g(x), \quad -\infty < x < \infty$$

where $g(x)$ is a continuous solution of the equation

$$g(x) = f_0(x) + \mu \int_{-\infty}^{\infty} dy g(y) g(x+1-y)$$

When $|x| \to \infty$, we can obtain in the same manner

$$f(x) < q f_0(x); \quad q > 1, \quad q = \text{const}$$

provided that the Fourier transform $F_0(k)$ of the function $f_0(x)$ is such, that $F_0(k) \to 0$ as $k \to 0$.

The sufficient condition for the series (4.1)-(4.5) to converge in the region $R_0 \leqslant R \leqslant R_1$ is, that the following integral converges:

$$\int_{-\infty}^{\infty} f_0(x) \left(\frac{R}{R_0}\right)^{-x} dx \leqslant c_1 < \infty$$

In particular, if

$$f_0(x) \leqslant \frac{c_0}{(1+x^2)^{\alpha/2}} \left(\frac{R_1}{R_0}\right)^x, \quad x \leqslant 0$$

$$f_0(x) \leqslant \frac{c_0}{(1+x^2)^{\alpha/2}}, \quad x > 0$$

$$0 < c_0 < \infty, \quad \alpha > 1$$

then the necessary requirements will be satisfied and the series will converge absolutely everywhere within the region $[R_0\ R_1]$. We should remember that a stronger condition is imposed on the asymptotic behaviour of the characteristic multipoles in the case of positive powers of the spherical radius R as $(N \to -\infty$ or $x \to -\infty)$. Therefore, the domain of applicability of the proposed expansions for a jet flow in a spherical layer may be narrower (according to the admissible form of the velocity profiles on the outer sphere, or relative to the number Re), than in the case of a boundary value problem outside the sphere.

REFERENCES

1. GOL'DSHTIK M.A. and YAVORSKII N.I., On submerged jets. PMM, 50, 4, 1986.
2. LANDAU L.D., On a new exact solution of the Navier-Stokes equations. Dokl. Akad. Nauk SSSR, 43, 7, 1944.
3. TSUKKER M.S., A twisted jet propagating in a space submerged in the same fluid. PMM, 19, 4, 1955.
4. TRIBEL H., Theory of Interpolation, Functional Spaces and Differential Operators. Mir, Moscow, 1980.
5. SEDOV L.I., Similarity and Dimensionality Methods in Mechanics. Nauka, Moscow, 1977.
6. SCHLICHTING H., Boundary Layer Theory. McGraw-Hill, New York, 1979.
7. GOL'DSHTIK M.A., Vortical Flows. Nauka, Novosibirsk, 1981.
8. KUZOV K., MELIKOV A. and LOZANOVA A., Macrostructure of a free turbulent jet with rectangular initial cross-section. Proceedings of the Fifth National Congress on Theoretical and Applied Mechanics, Varna, Sofiya, 1985.
9. KROTHAPALLI A., BAGANOFF D. and KARAMSCHETI K., On the mixing of a rectangular jet, J. Fluid Mech. 107, 1981.
10. MORRIS P.J., The spatial viscous instability of axisymmetric jets, J. Fluid Mech. 77, 3, 1976.
11. YAVORSKII N.I., A non-selfsimilar jet propagating in a bounded space. Problems in Thermal Physics and Physical Hydrodynamics, Inst. Teplofiziki, Akad. Nauk SSSR, Novosibirsk, 1985.
12. REYNOLDS A.J., Observation of a liquid-into-liquid jet, J. Fluid Mech. 14, 4, 1962.
13. LESSEN M. and SINGH P.J., The stability of axisymmetric free shear layers, J. Fluid Mech. 60, 3, 1973.

Translated by L.K.